Defects between Gapped Boundaries in $(2 + 1)D$
Topological Phases of Matter

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(2 + 1)$D$ topological phases

- (Non-abelian) bulk anyons often used for topological quantum computation (TQC)
- Difficult to realize
Introduction and Motivations

- $(2 + 1)D$ topological phases
  - (Non-abelian) bulk anyons often used for topological quantum computation (TQC)
  - Difficult to realize
- Want non-abelian objects arising from abelian materials → study gapped boundaries and defects
(2 + 1)$D$ topological phases
- (Non-abelian) bulk anyons often used for topological quantum computation (TQC)
- Difficult to realize
- Want non-abelian objects arising from abelian materials $\rightarrow$ study gapped boundaries and defects
- Gapped boundaries and boundary defects found in physical (FQH/SC/FM) systems (Lindner, Berg, Refael, Stern)

Introduction and Motivations

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Defects between gapped boundaries

JMM, January 2018
Review:

- Levin-Wen model
- Gapped boundaries, indecomposable modules, and Lagrangian algebras
- Condensation
Overview

- **Review:**
  - Levin-Wen model
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  - Condensation

- **Main Contributions:**
  - Boundary defects through multi-fusion category
  - Relation with bulk symmetry defects: crossed condensation
  - Braiding boundary defects
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- **Outlook**

A topological phase of matter is an equivalence class $\mathcal{H}$ of gapped Hamiltonians that realize a topological quantum field theory (TQFT) at low energy.

One family of such Hamiltonians is the Levin-Wen model.
The Levin-Wen model

- Input: trivalent lattice (e.g. honeycomb), unitary fusion category (UFC) $\mathcal{C}$
- Realizes TQFT given by Drinfeld center $\mathcal{Z}(\mathcal{C})$
  - $\mathcal{Z}(\mathcal{C})$ is a modular tensor category (MTC):
    - simple objects form anyon system (UFC + non-degenerate braiding)
The Levin-Wen model: Examples

- Example: $\mathcal{D}(\mathbb{Z}_p)$, the $\mathbb{Z}_p$ toric code
  - Anyons: $e^j m^k$, $0 \leq j, k \leq p - 1$, $e^{j_1} m^{k_1} \otimes e^{j_2} m^{k_2} \rightarrow e^{j_1 + j_2} m^{k_1 + k_2}$ (mod $p$)

- Example: $\mathcal{D}(S_3)$

**Table**: Fusion rules of $\mathcal{D}(S_3)$

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A gapped boundary is an equivalence class of gapped local (commuting) extensions of $H \in \mathcal{H}$ to the boundary.

Levin-Wen model: indecomposable (left) module category $\mathcal{M}$ of $\mathcal{C}$ (Kitaev and Kong):

- Category $\mathcal{M}$ with (left) $\mathcal{C}$-action: $\mathcal{C} \otimes \mathcal{M} \rightarrow \mathcal{M}$, associativity/unit constraints
- Not direct sum of other such categories
Theorem (Ostrik)

When $\mathcal{C} = \text{Rep}(G)$ or $\text{Vec}_G$ and $\mathcal{B}$ is a quantum double, the indecomposable modules $\mathcal{M}$ of $\mathcal{C} \leftrightarrow$ pairs $(K, \omega)$, $K \subseteq G$ (up to conjugation), $\omega \in H^2(K, \mathbb{C}^\times)$. 
Gapped boundaries: Lagrangian algebras

Definition

A Lagrangian algebra $\mathcal{A}$ in a MTC $\mathcal{B}$ is an object with a multiplication $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ such that:

1. $\mathcal{A}$ is commutative, i.e. $\mathcal{A} \otimes \mathcal{A} \xrightarrow{c_{\mathcal{A},\mathcal{A}}} \mathcal{A} \otimes \mathcal{A} \xrightarrow{m} \mathcal{A}$ equals $\mathcal{A} \otimes \mathcal{A} \xrightarrow{m} \mathcal{A}$, where $c_{\mathcal{A},\mathcal{A}}$ is the braiding in $\mathcal{B}$.

2. $\mathcal{A}$ is separable, i.e. the multiplication morphism $m$ admits a splitting $\mu : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ a morphism of $(\mathcal{A}, \mathcal{A})$-bimodules.

3. $\mathcal{A}$ is connected, i.e. $\text{Hom}_{\mathcal{B}}(1_{\mathcal{B}}, \mathcal{A}) = \mathbb{C}$

4. The Frobenius-Perron dimension (quantum dimension) of $\mathcal{A}$ is the square root of that of the MTC $\mathcal{B}$,

$$\text{FPdim}(\mathcal{A})^2 = \text{FPdim}(\mathcal{B}).$$ (1)
Gapped boundaries: Lagrangian algebras

Theorem (Davydov, Müger, Nikshych, Ostrik)
There exists a 1-1 correspondence between the indecomposable modules of $\mathcal{C}$ and the Lagrangian algebras of $\mathcal{B} = \mathcal{Z}(\mathcal{C})$.

Corollary
Gapped boundaries in anyon system $\mathcal{B} \leftrightarrow$ Lagrangian algebras $\mathcal{A}$ in $\mathcal{B}$

$\mathcal{A}$ is the collection of bulk anyons that condense to vacuum on the boundary
Gapped boundaries: Lagrangian algebras

Examples:

- $\mathcal{D}(\mathbb{Z}_p)$:

<table>
<thead>
<tr>
<th>$K_1 = {1}$</th>
<th>$A_1 = 1 + e + \ldots + e^{p-1}$</th>
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<tbody>
<tr>
<td>$K_2 = \mathbb{Z}_p$</td>
<td>$A_2 = 1 + m + \ldots + m^{p-1}$</td>
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- $\mathcal{D}(S_3)$:

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<th>$K_1 = {1}$</th>
<th>$A_1 = A + B + 2C$</th>
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<td>$K_2 = \mathbb{Z}_3$</td>
<td>$A_2 = A + B + 2F$</td>
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<td>$K_3 = \mathbb{Z}_2$</td>
<td>$A_3 = A + C + D$</td>
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<tr>
<td>$K_4 = S_3$</td>
<td>$A_4 = A + F + D$</td>
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Condensation

In general, bulk anyons may not condense to vacuum $\rightarrow$ condense to boundary excitations

**Definition**

Let $\mathcal{B}$ be a MTC, $\mathcal{A} \in \text{Obj} \mathcal{B}$ a Lagrangian algebra. The *quotient category* $\mathcal{B}/\mathcal{A}$ is the category s.t.

1. $\text{Obj} \mathcal{B}/\mathcal{A} = \text{Obj} \mathcal{B}$
2. $\text{Hom}_{\mathcal{B}/\mathcal{A}}(X, Y) = \text{Hom}_{\mathcal{B}}(X, \mathcal{A} \otimes Y)$.

The resulting category of excitations is the functor category $\text{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$ (a UFC)$^1$

---

$^1$In some cases, an *idempotent completion* may be necessary.
The condensation functor $F : \mathcal{B} \to \mathcal{B}/\mathcal{A}$ is a tensor functor.

Adjoint $I : \mathcal{B}/\mathcal{A} \to \mathcal{B}$ pulls excitation out of boundary, into bulk.
Examples:

- \( \mathcal{D}(\mathbb{Z}_p) \):
  \[ A_1 = 1 + e + \ldots + e^{p-1}, \quad e^a m^b \mapsto m^b, \]
  \[ \text{Fun}_C(\mathcal{M}_1, \mathcal{M}_1) = \{1, m, \ldots m^{p-1}\} \]

- \( \mathcal{D}(S_3) \):
  \[ A_2 = A + C + D: \]
  \[ A_1 = A + B + 2C: \]

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<td>( A + B )</td>
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<td>( D )</td>
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<td>( F, G, H )</td>
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<td>( E )</td>
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<td>( F, G, H )</td>
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Thus far, one boundary type at a time
Boundary defects

- Thus far, one boundary type at a time
- Multiple adjacent boundary types → boundary defects
Boundary defects

- Thus far, one boundary type at a time
- Multiple adjacent boundary types → boundary defects
- Boundary defects category:
  $\text{Fun}_\mathcal{C}(\mathcal{M}_i, \mathcal{M}_j)$ (Kitaev and Kong)
Boundary defects (finite groups)

Theorem (Ostrik)
Let $\mathcal{C} = \text{Rep}(G)$ or $\text{Vec}_G$. Suppose gapped boundaries $\mathcal{A}_1, \mathcal{A}_2 (\mathcal{M}_1, \mathcal{M}_2)$ are given by subgroups $K_1, K_2$ (and trivial cocycles). Then simple objects in $\text{Fun}_\mathcal{C}(\mathcal{M}_1, \mathcal{M}_2)$ are parametrized by pairs $(T, R)$, where $T = K_1 r_T K_2$ is a double coset, and $R$ is an irreducible representation of the stabilizer $(K_1, K_2)^{r_T} = K_1 \cap r_T K_2 r_T^{-1}$.

Theorem (Yamagami)
The quantum dimension of $(T, R)$ is

$$\text{FPdim}(T, R) = \frac{\sqrt{|K_1||K_2|}}{|K_1 \cap r_T K_2 r_T^{-1}|} \cdot \text{Dim}(R).$$  (2)
Boundary defects: Examples

Examples

- $\mathcal{D}(\mathbb{Z}_p)$: $K_1 = \{1\}, K_2 = \mathbb{Z}_p$. Single (simple) boundary defect of quantum dimension $\sqrt{p}$

- $\mathcal{D}(S_3)$:

<table>
<thead>
<tr>
<th>$\mathcal{A}_1$</th>
<th>$\mathcal{A}_2$</th>
<th>$\mathcal{A}_3$</th>
<th>$\mathcal{A}_4$</th>
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<tr>
<td>$\mathcal{A}_1 = A + B + 2C / K_1 = {1}$</td>
<td>Vec$_{S_3}$</td>
<td>${\sqrt{3}, \sqrt{3}}$</td>
<td>${\sqrt{2}, \sqrt{2}, \sqrt{2}}$</td>
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<tr>
<td>$\mathcal{A}_2 = A + B + 2F / K_2 = \mathbb{Z}_3$</td>
<td>${\sqrt{3}, \sqrt{3}}$</td>
<td>Vec$_{S_3}$</td>
<td>${\sqrt{6}}$</td>
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<tr>
<td>$\mathcal{A}_3 = A + C + D / K_3 = \mathbb{Z}_2$</td>
<td>${\sqrt{2}, \sqrt{2}, \sqrt{2}}$</td>
<td>${\sqrt{6}}$</td>
<td>Rep($S_3$)</td>
</tr>
<tr>
<td>$\mathcal{A}_4 = A + F + D / K_4 = S_3$</td>
<td>${\sqrt{6}}$</td>
<td>${\sqrt{2}, \sqrt{2}, \sqrt{2}}$</td>
<td>${\sqrt{3}, \sqrt{3}}$</td>
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Boundary defects: multi-fusion category

- $C_{ij} = \text{Fun}_C(M_i, M_j)$ is not a fusion category
- $\Gamma = \{C_{ij}\}$ (all possible excitations and boundary defects) is a multi-fusion category
  - $1$ is not simple
  - Can compute quantum dimensions, etc.
- For TQC: Can we obtain braiding?
- Solution: Examine bulk counterparts
Recall: condensation functor $F : \mathcal{B} = \mathcal{Z}(\mathcal{C}) \to \mathcal{B}/\mathcal{A} = \mathcal{C}_{ii}$
$= \text{Fun}_{\mathcal{C}}(\mathcal{M}_i, \mathcal{M}_i)$ and adjoint $I$

Want a similar construction for boundary defects
Bulk symmetry defects

- Symmetries of a MTC: $\text{Aut}^{br}(\mathcal{B})$
- Global symmetry group: $\rho : G \rightarrow \text{Aut}^{br}(\mathcal{B})$
- Bulk symmetry defects form a $G$-graded fusion category (Barkeshli, Bonderson, Cheng, Wang):

$$\mathcal{B}_G = \bigoplus_{g \in G} \mathcal{B}_g, \quad \mathcal{B}_0 = \mathcal{B} \quad (3)$$
**Bulk symmetry defects**

- Fusion of symmetry defects respects group multiplication:
  \[ a_g \otimes b_h \rightarrow c_{gh} \]
- **G-crossed braiding** (Barkeshli, Bonderson, Cheng, Wang):

\[
R^{a_g b_h} = \sum_{c, \mu, \nu} \sqrt{\frac{d_c}{d_a d_b}} \left[ R^{a_g b_h}_{c_{gh}} \right]_{\mu \nu}
\]

---

Figure: M. Barkeshli, P. Bonderson, M. Cheng, Z. Wang (2014)
Bulk symmetry defects: Examples

- \( \mathcal{O}(\mathbb{Z}_p) : G = \mathbb{Z}_2, e \leftrightarrow m, \mathcal{B}_1 = \{\tau_0, \ldots \tau_{p-1}\}, \dim \tau_i = \sqrt{p} \)
- \( \mathcal{O}(S_3) : G = \mathbb{Z}_2, C \leftrightarrow F, \mathcal{B}_1 = \{\sqrt{3}, \sqrt{3}, \sqrt{3}, \sqrt{3}, 2\sqrt{3}, 2\sqrt{3}\} \)
Suppose $B = \mathcal{Z}(\mathcal{C})$, $\rho : G \to \text{Aut}^{\text{br}}(B)$, $\mathcal{A}_i \in B$ a gapped boundary. Then $A_{j_g} := \rho_g(\mathcal{A}_i) \in B$ is a gapped boundary.
Crossed condensation

Suppose \( \mathcal{B} = \mathcal{Z}(\mathcal{C}) \), \( \rho : G \to \text{Aut}^\text{br}_\otimes(\mathcal{B}) \), \( \mathcal{A}_i \in \mathcal{B} \) a gapped boundary. Then \( \mathcal{A}_{j_g} := \rho_g(\mathcal{A}_i) \in \mathcal{B} \) is a gapped boundary. Recall:

**Definition**

Let \( \mathcal{B} \) be a MTC, \( \mathcal{A}_i \in \text{Obj} \mathcal{B} \) a Lagrangian algebra. The *quotient category* \( \mathcal{B}/\mathcal{A}_i \) is the category s.t.

1. \( \text{Obj} \mathcal{B}/\mathcal{A}_i = \text{Obj} \mathcal{B} \)
2. \( \text{Hom}_{\mathcal{B}/\mathcal{A}_i}(X, Y) = \text{Hom}_\mathcal{B}(X, \mathcal{A}_i \otimes Y) \).

Replace the MTC \( \mathcal{B} \) with the \( G \)-graded category \( \mathcal{B}_G \).

Result:

\( F : \mathcal{B}_G \to \mathcal{Q}(G, \mathcal{A}_i) = \bigoplus_{g \in G} C_{ij}^g = \bigoplus_{g \in G} \text{Fun}_\mathcal{C}(M_i, M_j^g) \) (4)

(C, Cheng, Wang)

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Crossed condensation

Suppose $B = Z(C)$, $\rho : G \to \text{Aut}^{br}(B)$, $A_i \in B$ a gapped boundary. Then $A_{j_g} := \rho_g(A_i) \in B$ is a gapped boundary. Recall:

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Replace the MTC $B$ with the $G$-graded category $B_G$. 

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Defects between gapped boundaries

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\]

(4)

(C, Cheng, Wang)
Crossed condensation

Crossed condensation functor:

\[ F : \mathcal{B}_G \to \mathcal{Q}(G, \mathcal{A}_i) = \bigoplus_{g \in G} C_{ijg} = \bigoplus_{g \in G} \text{Fun}_\mathcal{C}(\mathcal{M}_i, \mathcal{M}_{jg}) \]
Crossed condensation functor:

\[ F : \mathcal{B}_G \rightarrow \mathcal{Q}(G, \mathcal{A}_i) = \bigoplus_{g \in G} C_{ijg} = \bigoplus_{g \in G} \text{Fun}_C(\mathcal{M}_i, \mathcal{M}_{jg}) \]

Physical explanation:

Diagram:

\[ \mathcal{A}_i \quad Y_- \quad \cdots \quad Y_+ \]

\[ \mathcal{A}_i \quad X_{ij} \quad \mathcal{A}_j \]
Crossed condensation: Examples

- $\mathcal{D}(\mathbb{Z}_p)$: $B_1 = \{\tau_0, \ldots, \tau_{p-1}\}$, $C_{12} = \{\tau_{12}\}$, $\tau_i \mapsto \tau_{12}$
- $\mathcal{D}(S_3)$: $B_1 = \{\sqrt{3}, \sqrt{3}, \sqrt{3}, \sqrt{3}, 2\sqrt{3}, 2\sqrt{3}\} \rightarrow C_{12} = \{\sqrt{3}, \sqrt{3}\}$
**Theorem (C, Cheng, Wang)**

Let $\mathcal{A}_i$ and $\mathcal{A}_j$ be two Lagrangian algebras (gapped boundaries) in $\mathcal{B} = \mathbb{Z}(C)$, and let $\mathcal{M}_i$ and $\mathcal{M}_j$ be the corresponding indecomposable module categories. Suppose $\mathcal{A}_i$ and $\mathcal{A}_j$ are related by a global $G$ symmetry of $\mathcal{B}$. Then:

1. There is a projective $G$-crossed braiding of the boundary defects in $\mathcal{C}_{ij}$ with those in $\mathcal{C}_{ji}$, and with the boundary excitations in $\mathcal{C}_{jj}$.

2. There is a canonical choice of this braiding and a systematic method to compute the projective representation.

3. If all defects in $\mathcal{C}_{ji}$ are fixed by the action of $g \in G$, the projective $G$-crossed braiding is a projective braiding of boundary defects.

Braiding done in the bulk (through correspondence):

$$X_{ij} \otimes X_{ji} \rightarrow I(X_{ij}) \otimes I(X_{ji}) \xrightarrow{G^\times} \rho_1(I(X_{ji})) \otimes I(X_{ij}) \rightarrow \rho_1(X_{ji}) \otimes X_{ij}. \quad (5)$$
Braiding boundary defects: Examples

- $\mathcal{D}(\mathbb{Z}_2)$: get $\pi/16$ phase gate, Majorana zero mode braid statistics
- $\mathcal{D}(S_3)$: expect $SU(2)_4$ braiding, which would give universal TQC
Known:
- Gapped boundaries as indecomposable modules, Lagrangian algebras
- Boundary excitations, defects in multi-fusion category
- Bulk-edge correspondence for certain boundary defects, symmetry defects $\rightarrow$ braiding

Goal:
- Other boundary defects not covered by this correspondence?
- New symmetry in the bulk?
- Efficient encoding/gates for topological quantum computation?
Acknowledgments

- Special to Maissam Barkeshli, Shawn Cui, Cesar Galindo for answering many questions
- Many thanks to Prof. Michael Freedman and everyone at Station Q
- None of this would have been possible without the guidance and dedication of Prof. Zhenghan Wang
Thanks!

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Defects between gapped boundaries

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Unitary fusion categories (UFCs)

Some key properties of a UFC $\mathcal{C}$ over $\mathbb{C}$:

- **Monoidal structure:**
  - Tensor product $\otimes$: fusion
  - Tensor unit $1$: vacuum
  - Functorial associativity and unit isomorphisms, encoded by $F$ symbols

\[
\begin{align*}
  i & \quad j & \quad k \\
  m & \quad l & \quad n
\end{align*}
\]

\[
= \sum_{n} F_{l;nm}^{i,jk}
\]

Unitary fusion categories (UFCs)

Key properties: [Cont’d]

- **Semisimplicity**: All objects are direct sums of *simple objects*
  - Finite number of simple objects, $\mathbf{1}$ is simple
  - *Fusion rules*: $x \otimes y \to \bigoplus C N_{xy}^z$
- **$\mathbb{C}$-linear**: $\text{Hom}(x, y)$ is a $\mathbb{C}$-vector space for all $x, y \in \text{Obj}(C)$, $\otimes$ bilinear on morphisms

Examples: $\text{Vec}_G$, $\text{Rep}(G)$, ...
The Levin-Wen model

\( \mathcal{Z}(\mathcal{C}) \) is a *modular tensor category* (MTC):

- MTC is a UFC, simple objects form *anyon system*
- *Braiding* structure: \( \sigma_{ab} : a \otimes b \rightarrow b \otimes a \) for all \( a, b \) (\( R \) symbols)
- Non-degeneracy: only *transparent* anyon is unit

\[
\begin{align*}
R_{c}^{ab} & = a \quad b \\
\bigcirc \quad \bigcirc & = R \quad R \\
\]
Theorem (Fröhlich, Fuchs, Runkel, Schweigert)
\[ A \text{ is a commutative algebra in a MTC } B \text{ if and only if the object } A \text{ decomposes into simple objects as } A = \bigoplus_s n_s s, \text{ with } \theta_s = 1 \text{ (i.e. } s \text{ is bosonic) for all } s \text{ such that } n_s \neq 0. \]

Theorem (C, Cheng, Wang)
A commutative connected algebra \( A = \bigoplus_s n_s s \) with \( \text{FPdim}(A)^2 = \text{FPdim}(B) \) is a Lagrangian algebra in the MTC \( B \) if and only if the following inequality holds for all \( a, b \in \text{Obj}(B) \):

\[
n_a n_b \leq \sum_c N_{ab}^c n_c \tag{6}
\]

where \( N_{ab}^c \) are the coefficients given by the fusion rules of \( B \).