Defects between Gapped Boundaries in $(2+1)D$ Topological Phases of Matter

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Introduction and Motivations

- $(2+1)D$ topological phases
  - (Non-abelian) bulk anyons often used for topological quantum computation (TQC)
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  - (Non-abelian) bulk anyons often used for topological quantum computation (TQC)
  - Difficult to realize
- Want non-abelian objects arising from abelian materials $\rightarrow$ study gapped boundaries and defects
- Gapped boundaries and boundary defects found in physical (FQH/SC/FM) systems (Lindner, Berg, Refael, Stern)

Overview

- Review:
  - Levin-Wen model
  - Gapped boundaries, indecomposable modules, and Lagrangian algebras
  - Condensation

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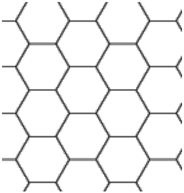
- Outlook
- (Ref. for gapped boundaries: C, Cheng, Wang, *Comm. Math. Phys.* (2017) 355: 645.)

The Levin-Wen model

- A *topological phase of matter* is an equivalence class $\mathcal{H}$ of gapped Hamiltonians that realize a topological quantum field theory (TQFT) at low energy
- One family of such Hamiltonians is the Levin-Wen model

The Levin-Wen model

- Input: trivalent lattice (e.g. honeycomb), unitary fusion category (UFC) $\mathcal{C}$
- Realizes TQFT given by Drinfeld center $\mathcal{Z}(\mathcal{C})$
  - $\mathcal{Z}(\mathcal{C})$ is a *modular tensor category* (MTC): simple objects form anyon system (UFC $+$ non-degenerate braiding)



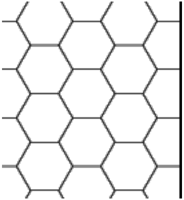
- Example: $\mathfrak{D}(\mathbb{Z}_p)$, the $\mathbb{Z}_p$ toric code
  - Anyons: $e^j m^k$, $0 \leq j, k \leq p-1$, $e^{j_1} m^{k_1} \otimes e^{j_2} m^{k_2} \to e^{j_1+j_2} m^{k_1+k_2}$ (mod p)
- Example: $\mathfrak{D}(S_3)$

Table: Fusion rules of $\mathfrak{D}(S_3)$

$\otimes$	A	B	C	D	E	F	G	H
A	A	B	C	D	E	F	G	H
B	B	A	C	E	D	F	G	H
C	C	C	$A \oplus B \oplus C$	$D \oplus E$	$D \oplus E$	$G \oplus H$	$F \oplus H$	$F \oplus G$
D	D	E	$D \oplus E$	$A \oplus C \oplus F$ $\oplus G \oplus H$	$B \oplus C \oplus F$ $\oplus G \oplus H$	$D \oplus E$	$D \oplus E$	$D \oplus E$
E	E	D	$D \oplus E$	$B \oplus C \oplus F$ $\oplus G \oplus H$	$A \oplus C \oplus F$ $\oplus G \oplus H$	$D \oplus E$	$D \oplus E$	$D \oplus E$
F	F	F	$G \oplus H$	$D \oplus E$	$D \oplus E$	$A \oplus B \oplus F$	$H \oplus C$	$G \oplus C$
G	G	G	$F \oplus H$	$D \oplus E$	$D \oplus E$	$H \oplus C$	$A \oplus B \oplus G$	$F \oplus C$
H	H	H	$F \oplus G$	$D \oplus E$	$D \oplus E$	$G \oplus C$	$F \oplus C$	$A \oplus B \oplus H$

Gapped boundaries: indecomposable modules

- A *gapped boundary* is an equivalence class of gapped local (commuting) extensions of $H \in \mathcal{H}$ to the boundary
- Levin-Wen model: *indecomposable (left) module* category $\mathcal{M}$ of $\mathcal{C}$ (Kitaev and Kong)
  - Category $\mathcal{M}$ with (left) $\mathcal{C}$-action: $\mathcal{C} \otimes \mathcal{M} \to \mathcal{M}$, associativity/unit constraints
  - Not direct sum of other such categories



Gapped boundaries: indecomposable modules

Theorem (Ostrik)

When $\mathcal{C} = \text{Rep}(G)$ or Vec_G and $\mathcal{B}$ is a quantum double, the indecomposable modules $\mathcal{M}$ of $\mathcal{C} \leftrightarrow$ pairs (K, ω), $K \subseteq G$ (up to conjugation), $\omega \in H^2(K, \mathbb{C}^\times)$.

Gapped boundaries: Lagrangian algebras

Definition

A *Lagrangian algebra* $\mathcal{A}$ in a MTC $\mathcal{B}$ is an object with a multiplication $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ such that:

1. $\mathcal{A}$ is *commutative*, i.e. $\mathcal{A} \otimes \mathcal{A} \xrightarrow{c_{\mathcal{A}\mathcal{A}}} \mathcal{A} \otimes \mathcal{A} \xrightarrow{m} \mathcal{A}$ equals $\mathcal{A} \otimes \mathcal{A} \xrightarrow{m} \mathcal{A}$, where $c_{\mathcal{A}\mathcal{A}}$ is the braiding in $\mathcal{B}$.
2. $\mathcal{A}$ is *separable*, i.e. the multiplication morphism m admits a splitting $\mu : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ a morphism of $(\mathcal{A}, \mathcal{A})$-bimodules.
3. $\mathcal{A}$ is *connected*, i.e. $\mathrm{Hom}_{\mathcal{B}}(\mathbf{1}_{\mathcal{B}}, \mathcal{A}) = \mathbb{C}$
4. The Frobenius-Perron dimension (quantum dimension) of $\mathcal{A}$ is the square root of that of the MTC $\mathcal{B}$,

$$\mathrm{FPdim}(\mathcal{A})^2 = \mathrm{FPdim}(\mathcal{B}). \tag{1}$$

Gapped boundaries: Lagrangian algebras

Theorem (Davydov, Müger, Nikshych, Ostrik)

There exists a 1-1 correspondence between the indecomposable modules of $\mathcal{C}$ and the Lagrangian algebras of $\mathcal{B} = \mathcal{Z}(\mathcal{C})$.

Corollary

Gapped boundaries in anyon system $\mathcal{B} \leftrightarrow$ Lagrangian algebras $\mathcal{A}$ in $\mathcal{B}$

$\mathcal{A}$ is the collection of bulk anyons that condense to vacuum on the boundary

Gapped boundaries: Lagrangian algebras

Examples:

- $\mathfrak{D}(\mathbb{Z}_p)$:

$K_1 = \{1\}$	$A_1 = 1 + e + .. + e^{p-1}$
$K_2 = \mathbb{Z}_p$	$\mathcal{A}_2 = 1 + m + ... + m^{p-1}$

- $\mathfrak{D}(S_3)$:

$K_1 = \{1\}$	$\mathcal{A}_1 = A + B + 2C$
$K_2 = \mathbb{Z}_3$	$\mathcal{A}_2 = A + B + 2F$
$K_3 = \mathbb{Z}_2$	$\mathcal{A}_3 = A + C + D$
$K_4 = S_3$	$\mathcal{A}_4 = A + F + D$

Condensation

In general, bulk anyons may not condense to vacuum $\rightarrow$ condense to *boundary excitations*

Definition

Let $\mathcal{B}$ be a MTC, $\mathcal{A} \in \mathrm{Obj}\,\mathcal{B}$ a Lagrangian algebra. The *quotient category* $\mathcal{B}/\mathcal{A}$ is the category s.t.

1. $\mathrm{Obj}\,\mathcal{B}/\mathcal{A} = \mathrm{Obj}\,\mathcal{B}$
2. $\mathrm{Hom}_{\mathcal{B}/\mathcal{A}}(X, Y) = \mathrm{Hom}_{\mathcal{B}}(X, \mathcal{A} \otimes Y)$.

The resulting category of excitations is the functor category $\mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$ (a UFC)[1]

[1] In some cases, an *idempotent completion* may be necessary.

Condensation

- The condensation functor $F : \mathcal{B} \to \mathcal{B}/\mathcal{A}$ is a tensor functor
- Adjoint $I : \mathcal{B}/\mathcal{A} \to \mathcal{B}$ pulls excitation out of boundary, into bulk

Condensation: Examples

Examples:

- $\mathfrak{D}(\mathbb{Z}_p)$:
  - $\mathcal{A}_1 = 1 + e + ... + e^{p-1}$, $e^a m^b \mapsto m^b$,
    $\mathrm{Fun}_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M}_1) = \{1, m, ... m^{p-1}\}$
- $\mathfrak{D}(S_3)$:
  - $\mathcal{A}_2 = A + C + D$:

$\boldsymbol{a}$	$\boldsymbol{F(a)}$
A	A
B	B
C	$A+B$
D	$B+F$
E	$B+F$
F, G, H	F

  - $\mathcal{A}_1 = A + B + 2C$:

$\boldsymbol{a}$	$\boldsymbol{F(a)}$
A	1
B	1
C	$2 \cdot 1$
D	$s + sr + sr^2$
E	$s + sr + sr^2$
F, G, H	$r + r^2$

Boundary defects

- Thus far, one boundary type at a time

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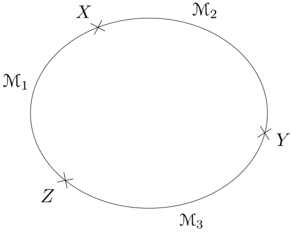
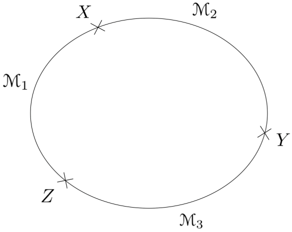
- Multiple adjacent boundary types $\rightarrow$ boundary defects



Boundary defects

- Thus far, one boundary type at a time
- Multiple adjacent boundary types $\rightarrow$ boundary defects
- Boundary defects category: $\mathrm{Fun}_{\mathcal{C}}(\mathcal{M}_i, \mathcal{M}_j)$ (Kitaev and Kong)



Boundary defects (finite groups)

Theorem (Ostrik)

Let $\mathcal{C} = \text{Rep}(G)$ or Vec_G. Suppose gapped boundaries $\mathcal{A}_1$, $\mathcal{A}_2$ ($\mathcal{M}_1$, M_2) are given by subgroups K_1, K_2 (and trivial cocycles). Then simple objects in $\text{Fun}_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M}_2)$ are parametrized by pairs (T, R), where $T = K_1 r_T K_2$ is a double coset, and R is an irreducible representation of the stabilizer $(K_1, K_2)^{r_T} = K_1 \cap r_T K_2 r_T^{-1}$.

Theorem (Yamagami)

The quantum dimension of (T, R) is

$$\text{FPdim}(T, R) = \frac{\sqrt{|K_1||K_2|}}{|K_1 \cap r_T K_2 r_T^{-1}|} \cdot \text{Dim}(R). \qquad (2)$$

Boundary defects: Examples

Examples

- $\mathfrak{D}(\mathbb{Z}_p)$: $K_1 = \{1\}$, $K_2 = \mathbb{Z}_p$. Single (simple) boundary defect of quantum dimension $\sqrt{p}$
- $\mathfrak{D}(S_3)$:

	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_4$
$\mathcal{A}_1 = A + B + 2C \ / \ K_1 = \{1\}$	Vec_{S_3}	$\left\{\sqrt{3}, \sqrt{3}\right\}$	$\left\{\sqrt{2}, \sqrt{2}, \sqrt{2}\right\}$	$\left\{\sqrt{6}\right\}$
$\mathcal{A}_2 = A + B + 2F \ / \ K_2 = \mathbb{Z}_3$	$\left\{\sqrt{3}, \sqrt{3}\right\}$	Vec_{S_3}	$\left\{\sqrt{6}\right\}$	$\left\{\sqrt{2}, \sqrt{2}, \sqrt{2}\right\}$
$\mathcal{A}_3 = A + C + D \ / \ K_3 = \mathbb{Z}_2$	$\left\{\sqrt{2}, \sqrt{2}, \sqrt{2}\right\}$	$\left\{\sqrt{6}\right\}$	$\mathrm{Rep}(S_3)$	$\left\{\sqrt{3}, \sqrt{3}\right\}$
$\mathcal{A}_4 = A + F + D \ / \ K_4 = S_3$	$\left\{\sqrt{6}\right\}$	$\left\{\sqrt{2}, \sqrt{2}, \sqrt{2}\right\}$	$\left\{\sqrt{3}, \sqrt{3}\right\}$	$\mathrm{Rep}(S_3)$

Boundary defects: multi-fusion category

- $\mathcal{C}_{ij} = \mathsf{Fun}_{\mathcal{C}}(\mathcal{M}_i, \mathcal{M}_j)$ is not a fusion category
- $\Gamma = \{\mathcal{C}_{ij}\}$ (all possible excitations and boundary defects) is a *multi-fusion* category
  - $\mathbf{1}$ is not simple
  - Can compute quantum dimensions, etc.
- For TQC: Can we obtain braiding?
- Solution: Examine bulk counterparts

Condensation

- Recall: condensation functor $F : \mathcal{B} = \mathcal{Z}(\mathcal{C}) \to \mathcal{B}/\mathcal{A} = \mathcal{C}_{ii}$ $= \mathrm{Fun}_{\mathcal{C}}(\mathcal{M}_i, \mathcal{M}_i)$ and adjoint I
- Want a similar construction for boundary defects

Bulk symmetry defects

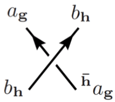
- Symmetries of a MTC: $\mathrm{Aut}_{\otimes}^{\mathrm{br}}(\mathcal{B})$
- Global symmetry group: $\rho : G \to \mathrm{Aut}_{\otimes}^{\mathrm{br}}(\mathcal{B})$
- Bulk symmetry defects form a G-graded fusion category (Barkeshli, Bonderson, Cheng, Wang):

$$\mathcal{B}_G = \oplus_{g \in G} \mathcal{B}_g, \qquad \mathcal{B}_0 = \mathcal{B} \tag{3}$$

Bulk symmetry defects

- Fusion of symmetry defects respects group multiplication: $a_g \otimes b_h \to c_{gh}$
- G-crossed braiding (Barkeshli, Bonderson, Cheng, Wang):

$R^{a_{\mathbf{g}} b_{\mathbf{h}}} =$



$= \sum_{c,\mu,\nu} \sqrt{\frac{d_c}{d_a d_b}} \left[R^{a_{\mathbf{g}} b_{\mathbf{h}}}_{c_{\mathbf{gh}}} \right]_{\mu\nu}$

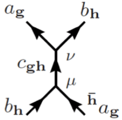


Figure: M. Barkeshli, P. Bonderson, M. Cheng, Z. Wang (2014)

Bulk symmetry defects: Examples

- $\mathfrak{D}(\mathbb{Z}_p)$: $G = \mathbb{Z}_2$, $e \leftrightarrow m$, $\mathcal{B}_1 = \{\tau_0, ...\tau_{p-1}\}$, $\dim \tau_i = \sqrt{p}$
- $\mathfrak{D}(S_3)$: $G = \mathbb{Z}_2$, $C \leftrightarrow F$, $\mathcal{B}_1 = \{\sqrt{3}, \sqrt{3}, \sqrt{3}, \sqrt{3}, 2\sqrt{3}, 2\sqrt{3}\}$

Crossed condensation

Suppose $\mathcal{B} = \mathcal{Z}(\mathcal{C})$, $\rho : G \to \mathrm{Aut}_{\otimes}^{\mathrm{br}}(\mathcal{B})$, $\mathcal{A}_i \in \mathcal{B}$ a gapped boundary.
Then $\mathcal{A}_{j_g} := \rho_g(\mathcal{A}_i) \in \mathcal{B}$ is a gapped boundary.

Crossed condensation

Suppose $\mathcal{B} = \mathcal{Z}(\mathcal{C})$, $\rho : G \to \mathrm{Aut}_{\otimes}^{\mathrm{br}}(\mathcal{B})$, $\mathcal{A}_i \in \mathcal{B}$ a gapped boundary. Then $\mathcal{A}_{j_g} := \rho_g(\mathcal{A}_i) \in \mathcal{B}$ is a gapped boundary. Recall:

Definition

Let $\mathcal{B}$ be a MTC, $\mathcal{A}_i \in \mathrm{Obj}\,\mathcal{B}$ a Lagrangian algebra. The *quotient category* $\mathcal{B}/\mathcal{A}_i$ is the category s.t.

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Replace the MTC $\mathcal{B}$ with the G-graded category $\mathcal{B}_G$.

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Replace the MTC $\mathcal{B}$ with the G-graded category $\mathcal{B}_G$. Result:

$$F : \mathcal{B}_G \to \mathcal{Q}(G, \mathcal{A}_i) = \oplus_{g \in G} \mathcal{C}_{ij_g} = \oplus_{g \in G} \mathrm{Fun}_{\mathcal{C}}(\mathcal{M}_i, \mathcal{M}_{j_g}) \tag{4}$$

(C, Cheng, Wang)

Crossed condensation

Crossed condensation functor:

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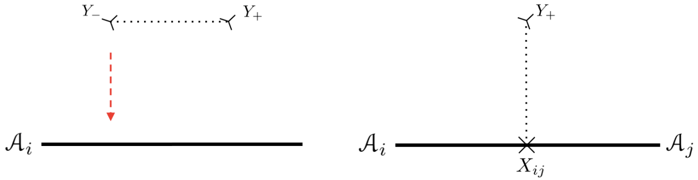
Physical explanation:



Crossed condensation: Examples

- $\mathfrak{D}(\mathbb{Z}_p)$: $\mathcal{B}_1 = \{\tau_0, ...\tau_{p-1}\}$, $\mathcal{C}_{12} = \{\tau_{12}\}$, $\tau_i \mapsto \tau_{12}$
- $\mathfrak{D}(S_3)$: $\mathcal{B}_1 = \{\sqrt{3}, \sqrt{3}, \sqrt{3}, \sqrt{3}, 2\sqrt{3}, 2\sqrt{3}\} \to \mathcal{C}_{12} = \{\sqrt{3}, \sqrt{3}\}$

Braiding boundary defects

Theorem (C, Cheng, Wang)

Let $\mathcal{A}_i$ and $\mathcal{A}_j$ be two Lagrangian algebras (gapped boundaries) in $\mathcal{B} = \mathcal{Z}(\mathcal{C})$, and let $\mathcal{M}_i$ and $\mathcal{M}_j$ be the corresponding indecomposable module categories. Suppose $\mathcal{A}_i$ and $\mathcal{A}_j$ are related by a global G symmetry of $\mathcal{B}$. Then:

1. There is a projective G-crossed braiding of the boundary defects in $\mathcal{C}_{ij}$ with those in $\mathcal{C}_{ji}$, and with the boundary excitations in $\mathcal{C}_{jj}$.
2. There is a canonical choice of this braiding and a systematic method to compute the projective representation.
3. If all defects in $\mathcal{C}_{ji}$ are fixed by the action of $g \in G$, the projective G-crossed braiding is a projective braiding of boundary defects.

Braiding done in the bulk (through correspondence):

$$X_{ij} \otimes X_{ji} \to I(X_{ij}) \otimes I(X_{ji}) \xrightarrow{G^{\times}} \rho_1(I(X_{ji})) \otimes I(X_{ij}) \to \rho_1(X_{ji}) \otimes X_{ij}. \quad (5)$$

Braiding boundary defects: Examples

- $\mathfrak{D}(\mathbb{Z}_2)$: get $\pi/16$ phase gate, Majorana zero mode braid statistics
- $\mathfrak{D}(S_3)$: expect $\mathrm{SU}(2)_4$ braiding, which would give universal TQC

Outlook

- Known:
  - Gapped boundaries as indecomposable modules, Lagrangian algebras
  - Boundary excitations, defects in multi-fusion category
  - Bulk-edge correspondence for certain boundary defects, symmetry defects $\rightarrow$ braiding
- Goal:
  - Other boundary defects not covered by this correspondence?
  - New symmetry in the bulk?
  - Efficient encoding/gates for topological quantum computation?

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- None of this would have been possible without the guidance and dedication of Prof. Zhenghan Wang

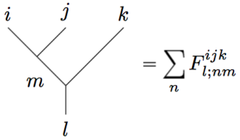
Thanks!

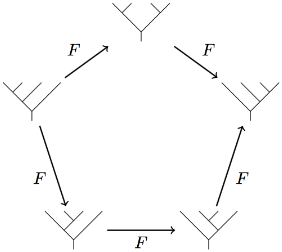
Backup slides

Unitary fusion categories (UFCs)

Some key properties of a UFC $\mathcal{C}$ over $\mathbb{C}$:

- *Monoidal* structure:
  - Tensor product $\otimes$: fusion
  - Tensor unit $\mathbf{1}$: vacuum
  - Functorial associativity and unit isomorphisms, encoded by F symbols





Figures: Z. Wang, *Topological Quantum Computation* (2010)

Unitary fusion categories (UFCs)

Key properties: [Cont'd]

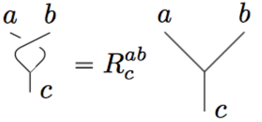
- *Semisimplicity*: All objects are direct sums of *simple objects*
  - Finite number of simple objects, $\mathbf{1}$ is simple
  - *Fusion rules*: $x \otimes y \to \oplus_C N_{xy}^z z$
- $\mathbb{C}$-*linear*: $\mathrm{Hom}(x, y)$ is a $\mathbb{C}$-vector space for all $x, y \in \mathrm{Obj}(\mathcal{C})$, $\otimes$ bilinear on morphisms

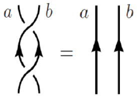
Examples: Vec_G, $\mathrm{Rep}(G)$, ...

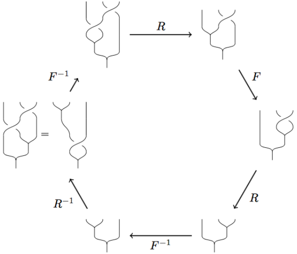
The Levin-Wen model

$\mathcal{Z}(\mathcal{C})$ is a *modular tensor category* (MTC):

- MTC is a UFC, simple objects form *anyon system*
- *Braiding* structure: $\sigma_{ab} : a \otimes b \to b \otimes a$ for all a, b (R symbols)
- Non-degeneracy: only *transparent* anyon is unit







Figures: Z. Wang, *Topological Quantum Computation* (2010)

Gapped boundaries: Lagrangian algebras

Theorem (Fröhlich, Fuchs, Runkel, Schweigert)

$\mathcal{A}$ is a commutative algebra in a MTC $\mathcal{B}$ if and only if the object $\mathcal{A}$ decomposes into simple objects as $\mathcal{A} = \oplus_s n_s s$, with $\theta_s = 1$ (i.e. s is bosonic) for all s such that $n_s \neq 0$.

Theorem (C, Cheng, Wang)

A commutative connected algebra $\mathcal{A} = \oplus_s n_s s$ with $\text{FPdim}(\mathcal{A})^2 = \text{FPdim}(\mathcal{B})$ is a Lagrangian algebra in the MTC $\mathcal{B}$ if and only if the following inequality holds for all $a, b \in \text{Obj}(\mathcal{B})$:

$$n_a n_b \leq \sum_c N_{ab}^c n_c \tag{6}$$

where N_{ab}^c are the coefficients given by the fusion rules of $\mathcal{B}$.