Gapped Boundaries and Defects in $(2+1)D$ Topological Phases

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  - (Non-abelian) bulk anyons often used for topological quantum computation (TQC)
  - Difficult to realize

Introduction and Motivations

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- Want non-abelian objects arising from abelian materials $\rightarrow$ study gapped boundaries and defects
- Gapped boundaries and boundary defects found in physical (FQH/SC/FM) systems (Lindner, Berg, Refael, Stern)

Overview

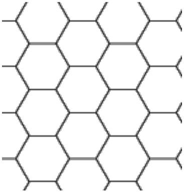
- Tensor categories and the Levin-Wen model
- Gapped boundaries, indecomposable modules, and Lagrangian algebras
- Ground state degeneracy, boundary excitations, condensation
- Boundary defects
- Bulk symmetry defects
- Crossed condensation
- Braiding boundary defects
- Outlook

The Levin-Wen model

- A *topological phase of matter* is an equivalence class $\mathcal{H}$ of gapped Hamiltonians that realize a topological quantum field theory at low energy
- One family of such Hamiltonians is the Levin-Wen model

The Levin-Wen model

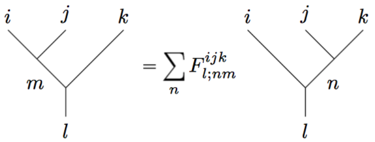
- Input: trivalent lattice (e.g. honeycomb)
- Each edge: element from a *unitary fusion category* (UFC) $\mathcal{C}$

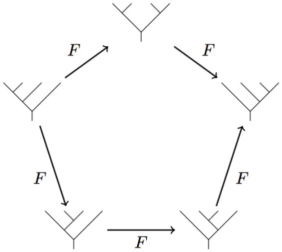


Unitary fusion categories (UFCs)

Some key properties of a UFC $\mathcal{C}$ over $\mathbb{C}$:

- *Monoidal* structure:
  - Tensor product $\otimes$: fusion
  - Tensor unit $\mathbf{1}$: vacuum
  - Functorial associativity and unit isomorphisms, encoded by F symbols





Figures: Z. Wang, *Topological Quantum Computation* (2010)

Unitary fusion categories (UFCs)

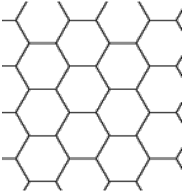
Key properties: [Cont'd]

- *Semisimplicity*: All objects are direct sums of *simple objects*
  - Finite number of simple objects, $\mathbf{1}$ is simple
  - *Fusion rules*: $x \otimes y \to \oplus_C N_{xy}^z z$
- $\mathbb{C}$-*linear*: $\text{Hom}(x, y)$ is a $\mathbb{C}$-vector space for all $x, y \in \text{Obj}(\mathcal{C})$, $\otimes$ bilinear on morphisms

Examples: Vec_G, $\text{Rep}(G)$, ...

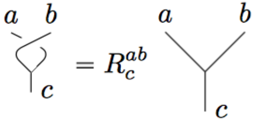
The Levin-Wen model

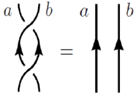
- Hamiltonian: vertex (charge conservation) and plaquette (zero-flux) constraints
- Realizes TQFT given by Drinfeld center $\mathcal{Z}(\mathcal{C})$

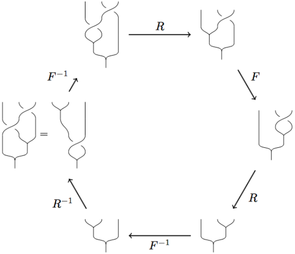


$\mathcal{Z}(\mathcal{C})$ is a *modular tensor category* (MTC):

- MTC is a UFC, simple objects form *anyon system*
- *Braiding* structure: $\sigma_{ab} : a \otimes b \to b \otimes a$ for all a, b (R symbols)
- Non-degeneracy: only *transparent* anyon is unit







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The Levin-Wen model

Theorem

If $\mathcal{C} = \text{Rep}(G)$ or Vec_G, $\mathcal{B} = \mathcal{Z}(\mathcal{C})$, anyon labels in $\mathcal{B}$ parameterized by (C, π): conjugacy class and irrep of $E(C)$

The Levin-Wen model

- Example: $\mathfrak{D}(\mathbb{Z}_p)$, the $\mathbb{Z}_p$ toric code
  - Anyons: $e^j m^k$, $0 \leq j, k \leq p-1$
  - Fusion rules: $e^{j_1} m^{k_1} \otimes e^{j_2} m^{k_2} \to e^{j_1+j_2} m^{k_1+k_2}$ (mod p)
  - Braiding: $R^{e^j m^k} = e^{2\pi ijk/p}$

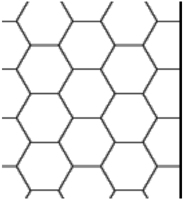
- Example: $\mathfrak{D}(S_3)$

Table: Fusion rules of $\mathfrak{D}(S_3)$

$\otimes$	A	B	C	D	E	F	G	H
A	A	B	C	D	E	F	G	H
B	B	A	C	E	D	F	G	H
C	C	C	$A \oplus B \oplus C$	$D \oplus E$	$D \oplus E$	$G \oplus H$	$F \oplus H$	$F \oplus G$
D	D	E	$D \oplus E$	$A \oplus C \oplus F \oplus G \oplus H$	$B \oplus C \oplus F \oplus G \oplus H$	$D \oplus E$	$D \oplus E$	$D \oplus E$
E	E	D	$D \oplus E$	$B \oplus C \oplus F \oplus G \oplus H$	$A \oplus C \oplus F \oplus G \oplus H$	$D \oplus E$	$D \oplus E$	$D \oplus E$
F	F	F	$G \oplus H$	$D \oplus E$	$D \oplus E$	$A \oplus B \oplus F$	$H \oplus C$	$G \oplus C$
G	G	G	$F \oplus H$	$D \oplus E$	$D \oplus E$	$H \oplus C$	$A \oplus B \oplus G$	$F \oplus C$
H	H	H	$F \oplus G$	$D \oplus E$	$D \oplus E$	$G \oplus C$	$F \oplus C$	$A \oplus B \oplus H$

Gapped boundaries: indecomposable modules

- A *gapped boundary* is an equivalence class of gapped local (commuting) extensions of $H \in \mathcal{H}$ to the boundary
- Levin-Wen model: *indecomposable (left) module* category $\mathcal{M}$ of $\mathcal{C}$ (Kitaev and Kong)
  - Category $\mathcal{M}$ with (left) $\mathcal{C}$-action: $\mathcal{C} \otimes \mathcal{M} \to \mathcal{M}$, associativity/unit constraints
  - Not direct sum of other such categories



Gapped boundaries: indecomposable modules

Theorem (Ostrik)

When $\mathcal{C} = \mathrm{Rep}(G)$ or Vec_G and $\mathcal{B}$ is a quantum double, the indecomposable modules $\mathcal{M}$ of $\mathcal{C} \leftrightarrow$ pairs (K, ω), $K \subseteq G$ (up to conjugation), $\omega \in H^2(K, \mathbb{C}^\times)$.

Gapped boundaries: indecomposable modules

- Example: $\mathfrak{D}(\mathbb{Z}_p)$

Gapped boundaries: Lagrangian algebras

Definition

A *Lagrangian algebra* $\mathcal{A}$ in a MTC $\mathcal{B}$ is an algebra with a multiplication $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ such that:

1. $\mathcal{A}$ is *commutative*, i.e. $\mathcal{A} \otimes \mathcal{A} \xrightarrow{c_{\mathcal{A}\mathcal{A}}} \mathcal{A} \otimes \mathcal{A} \xrightarrow{m} \mathcal{A}$ equals $\mathcal{A} \otimes \mathcal{A} \xrightarrow{m} \mathcal{A}$, where $c_{\mathcal{A}\mathcal{A}}$ is the braiding in $\mathcal{B}$.
2. $\mathcal{A}$ is *separable*, i.e. the multiplication morphism m admits a splitting $\mu : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ a morphism of $(\mathcal{A}, \mathcal{A})$-bimodules.
3. $\mathcal{A}$ is *connected*, i.e. $\mathrm{Hom}_{\mathcal{B}}(\mathbf{1}_{\mathcal{B}}, \mathcal{A}) = \mathbb{C}$
4. The Frobenius-Perron dimension (a.k.a. quantum dimension) of $\mathcal{A}$ is the square root of that of the MTC $\mathcal{B}$,

$$\mathrm{FPdim}(\mathcal{A})^2 = \mathrm{FPdim}(\mathcal{B}). \tag{1}$$

Gapped boundaries: Lagrangian algebras

Theorem (Davydov, Müger, Nikshych, Ostrik)

There exists a 1-1 correspondence between the indecomposable modules of $\mathcal{C}$ and the Lagrangian algebras of $\mathcal{B} = \mathcal{Z}(\mathcal{C})$.

Corollary

Gapped boundaries in anyon system $\mathcal{B} \leftrightarrow$ Lagrangian algebras $\mathcal{A}$ in $\mathcal{B}$

$\mathcal{A}$ is the collection of bulk bosonic anyons that condense to vacuum on the boundary

Gapped boundaries: Lagrangian algebras

Theorem (Fröhlich, Fuchs, Runkel, Schweigert)

$\mathcal{A}$ is a commutative algebra in a MTC $\mathcal{B}$ if and only if the object $\mathcal{A}$ decomposes into simple objects as $\mathcal{A} = \oplus_s n_s s$, with $\theta_s = 1$ (i.e. s is bosonic) for all s such that $n_s \neq 0$.

Theorem (C, Cheng, Wang)

A commutative connected algebra $\mathcal{A} = \oplus_s n_s s$ with $\text{FPdim}(\mathcal{A})^2 = \text{FPdim}(\mathcal{B})$ is a Lagrangian algebra in the MTC $\mathcal{B}$ if and only if the following inequality holds for all $a, b \in \text{Obj}(\mathcal{B})$:

$$n_a n_b \leq \sum_c N_{ab}^c n_c \tag{2}$$

where N_{ab}^c are the coefficients given by the fusion rules of $\mathcal{B}$.

Gapped boundaries: Lagrangian algebras

- Example: Lagrangian algebras in $\mathfrak{D}(\mathbb{Z}_p)$

- Example: Lagrangian algebras in $\mathfrak{D}(S_3)$

Table: Fusion rules of $\mathfrak{D}(S_3)$

$\otimes$	A	B	C	D	E	F	G	H
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Ground state degeneracy

- $\mathcal{A}$ is collection of bulk anyons that *condense to vacuum*
- Algebraic model: n holes on a plane

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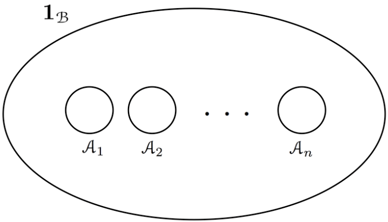

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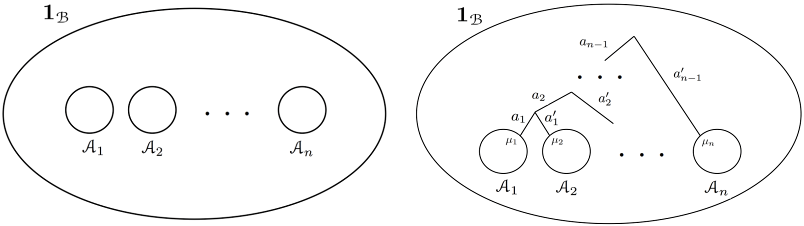
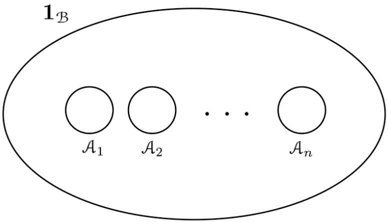
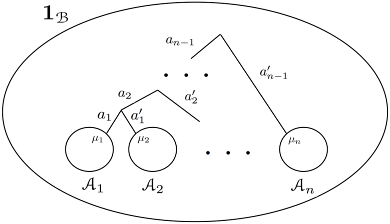


Ground state degeneracy

- $\mathcal{A}$ is collection of bulk anyons that *condense to vacuum*
- Algebraic model: n holes on a plane

$$\text{G.S.} = \text{Hom}(\mathbf{1}_B, \mathcal{A}_1 \otimes \mathcal{A}_2 \otimes ... \otimes \mathcal{A}_n) \tag{3}$$





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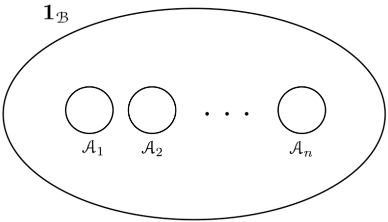

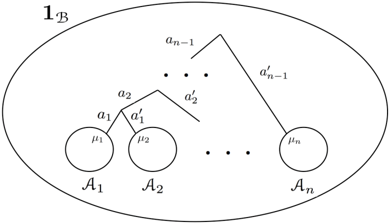

- $n = 2$ is used for a qudit encoding (C, Cheng, Wang)

Ground state degeneracy

- Example: $\mathfrak{D}(\mathbb{Z}_p)$

Condensation

In general, bulk anyons may not condense to vacuum $\rightarrow$ become *boundary excitations*

Definition

Let $\mathcal{B}$ be a MTC, $\mathcal{A} \in \operatorname{Obj} \mathcal{B}$ a Lagrangian algebra. The *quotient category* $\mathcal{B}/\mathcal{A}$ is the category s.t.

1. $\operatorname{Obj} \mathcal{B}/\mathcal{A} = \operatorname{Obj} \mathcal{B}$
2. $\operatorname{Hom}_{\mathcal{B}/\mathcal{A}}(X, Y) = \operatorname{Hom}_{\mathcal{B}}(X, \mathcal{A} \otimes Y)$.

The resulting category of excitations is the functor category $\operatorname{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$ (a UFC)

Condensation

- The condensation functor $F : \mathcal{B} \to \mathcal{B}/\mathcal{A}$ is a tensor functor
- Adjoint $I : \mathcal{B}/\mathcal{A} \to \mathcal{B}$ pulls excitation out of boundary, into bulk

Condensation

- Example: $\mathfrak{D}(\mathbb{Z}_p)$

Condensation

- Example: $\mathfrak{D}(S_3)$

Review

- Levin-Wen model: UFC $\mathcal{C} \to$ MTC $\mathcal{B} = \mathcal{Z}(\mathcal{C})$
- Gapped boundaries as indecomposable modules or Lagrangian algebras
- Ground state degeneracy
- Condensation and boundary excitations

Boundary defects

- Thus far, one boundary type per hole

Boundary defects

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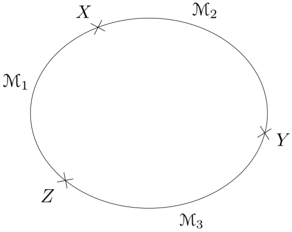
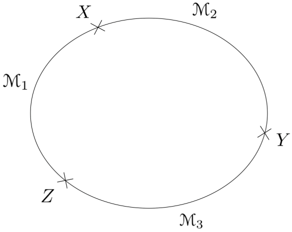
- Multiple boundary types per hole → boundary defects



Boundary defects

- Thus far, one boundary type per hole
- Multiple boundary types per hole $\rightarrow$ boundary defects
- Boundary defects category: $\text{Fun}_{\mathcal{C}}(\mathcal{M}_i, \mathcal{M}_j)$ (Kitaev and Kong)



Boundary defects

Theorem (Ostrik)

Let $\mathcal{C} = \text{Rep}(G)$ or Vec_G. Suppose gapped boundaries $\mathcal{A}_1$, $\mathcal{A}_2$ ($\mathcal{M}_1$, M_2) are given by subgroups K_1, K_2 (and trivial cocycles). Then simple objects in $\text{Fun}_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M}_2)$ are parametrized by pairs (T, R), where $T = K_1 r_T K_2$ is a double coset, and R is an irreducible representation of the stabilizer $(K_1, K_2)^{r_T} = K_1 \cap r_T K_2 r_T^{-1}$.

Theorem (Yamagami)

The quantum dimension of (T, R) is

$$\text{FPdim}(T, R) = \frac{\sqrt{|K_1||K_2|}}{|K_1 \cap r_T K_2 r_T^{-1}|} \cdot \text{Dim}(R). \quad (4)$$

Boundary defects

- Example: $\mathfrak{D}(\mathbb{Z}_p)$

Boundary defects

- Example: $\mathfrak{D}(S_3)$

Boundary defects

- $\mathcal{C}_{ij} = \mathsf{Fun}_{\mathcal{C}}(\mathcal{M}_i, \mathcal{M}_j)$ is not a fusion category
- $\Gamma = \{\mathcal{C}_{ij}\}$ (all possible excitations and boundary defects) is a *multi-fusion* category
  - $\mathbf{1}$ is not simple
  - Can compute quantum dimensions, etc.
- For TQC: Can we obtain braiding?
- Solution: Examine bulk counterparts

Condensation

- Recall: condensation functor $F : \mathcal{B} = \mathcal{Z}(\mathcal{C}) \to \mathcal{B}/\mathcal{A} = \mathcal{C}_{ii}$ $= \text{Fun}_{\mathcal{C}}(\mathcal{M}_i, \mathcal{M}_i)$ and adjoint I
- Want a similar construction for boundary defects

Bulk symmetry defects

- Symmetries of a MTC: $\mathrm{Aut}_{\otimes}^{\mathrm{br}}(\mathcal{B})$
- Symmetry: $\rho : G \to \mathrm{Aut}_{\otimes}^{\mathrm{br}}(\mathcal{B})$
- Bulk symmetry defects form a G-graded fusion category (Barkeshli, Bonderson, Cheng, Wang):

$$\mathcal{B}_G = \oplus_G \mathcal{B}_g, \qquad \mathcal{B}_0 = \mathcal{B} \tag{5}$$

Bulk symmetry defects

- Fusion of symmetry defects respects group multiplication: $a_g \otimes b_h \to c_{gh}$
- G-crossed braiding (Barkeshli, Bonderson, Cheng, Wang):

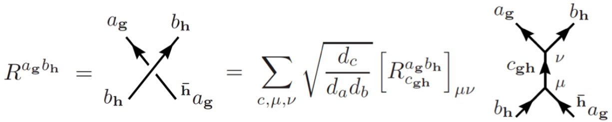


Figure: M. Barkeshli, P. Bonderson, M. Cheng, Z. Wang (2014)

Bulk symmetry defects

- Example: $\mathfrak{D}(\mathbb{Z}_p)$

Crossed condensation

Suppose $\mathcal{B} = \mathcal{Z}(\mathcal{C})$, $\rho : G \to \text{Aut}^{\text{br}}_{\otimes}(\mathcal{B})$, $\mathcal{A}_i \in \mathcal{B}$ a gapped boundary.
Then $\mathcal{A}_{j_g} := \rho_g(\mathcal{A}) \in \mathcal{B}$ is a gapped boundary.

Crossed condensation

Suppose $\mathcal{B} = \mathcal{Z}(\mathcal{C})$, $\rho : G \to \mathrm{Aut}_{\otimes}^{\mathrm{br}}(\mathcal{B})$, $\mathcal{A}_i \in \mathcal{B}$ a gapped boundary. Then $\mathcal{A}_{j_g} := \rho_g(\mathcal{A}) \in \mathcal{B}$ is a gapped boundary. Recall:

Definition

Let $\mathcal{B}$ be a MTC, $\mathcal{A}_i \in \mathrm{Obj}\,\mathcal{B}$ a Lagrangian algebra. The *quotient category* $\mathcal{B}/\mathcal{A}_i$ is the category s.t.

1. $\mathrm{Obj}\,\mathcal{B}/\mathcal{A}_i = \mathrm{Obj}\,\mathcal{B}$
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Replace the MTC $\mathcal{B}$ with the G-graded category $\mathcal{B}_G$.

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Replace the MTC $\mathcal{B}$ with the G-graded category $\mathcal{B}_G$. Result:

$$F : \mathcal{B}_G \to \mathcal{Q}(G, \mathcal{A}_i) = \oplus_{g\in G}\mathcal{C}_{ij_g} = \oplus_{g\in G}\,\mathrm{Fun}_{\mathcal{C}}(\mathcal{M}_i, \mathcal{M}_{j_g}) \qquad (6)$$

(C, Cheng, Wang)

Crossed condensation

Crossed condensation functor:

$$F : \mathcal{B}_G \to \mathcal{Q}(G, \mathcal{A}_i) = \oplus_{g \in G} \mathcal{C}_{ij_g} = \oplus_{g \in G} \operatorname{Fun}_{\mathcal{C}}(\mathcal{M}_i, \mathcal{M}_{j_g})$$

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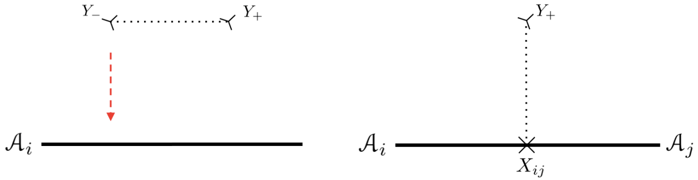
Physical explanation:



Crossed condensation

- Example: $\mathfrak{D}(\mathbb{Z}_p)$

Crossed condensation

- Example: $\mathfrak{D}(S_3)$

Braiding boundary defects

Theorem (C, Cheng, Wang)

Let $\mathcal{M}_i$, $\mathcal{M}_j$ be indecomposable module categories of $\mathcal{C}$. Suppose:

1. $\mathcal{C}_{ii} = \mathcal{C}_{jj}$ as fusion categories, and
2. $\mathcal{C}_{ij}$ is an invertible $\mathcal{C}_{ii} - \mathcal{C}_{jj}$ bimodule.

Then the boundary defects in $\mathcal{C}_{ij}$ can be projectively braided with those in $\mathcal{C}_{ji}$, and with the boundary excitations in $\mathcal{C}_{jj}$. Furthermore, there is a canonical choice of this braiding and a systematic method to compute the projective representation.

Braiding done in the bulk (through correspondence):

$$X_{ij} \otimes X_{ji} \to I(X_{ij}) \otimes I(X_{ji}) \xrightarrow{G^{\times}} \rho_1(I(X_{ji})) \otimes I(X_{ij}) \to \rho_1(X_{ji}) \otimes X_{ij}. \quad (7)$$

Braiding boundary defects

- Example: $\mathfrak{D}(\mathbb{Z}_p)$

Outlook

- Known:
  - Gapped boundaries as indecomposable modules, Lagrangian algebras
  - Boundary excitations, defects in multi-fusion category
  - Bulk-edge correspondence for certain boundary defects, symmetry defects $\rightarrow$ braiding
- Goal:
  - Other boundary defects not covered by this correspondence?
  - New symmetry in the bulk?